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Confined quantum fields under the influence of a uniform magnetic field

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Abstract

We investigate the influence of a uniform magnetic field on the zero-point energy of charged fields of two types, namely, a massive charged scalar field under Dirichlet boundary conditions and a massive fermion field under MIT boundary conditions. For the first, exact results are obtained, in terms of exponentially convergent functions, and for the second, the limits for small and for large masses are analytically obtained also. Coincidence with previously known, partial results serves as a check of the procedure. For the general case in the second situation—a rather involved one—a precise numerical analysis is performed.

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1. Introduction

The Casimir effect [1] is a very fundamental feature common to all quantum field theories, which arises in particular, as is well known, when there is a departure from the topology of the ordinary flat spacetime towards non-trivial topologies. It has been studied intensively in recent years due to its importance in elementary particle physics, cosmology and condensed matter physics—see [2] for a recent review on the theoretical and experimental aspects of this effect, and also [3].

Whenever we deal with a confined charged quantum field, it is a natural question to ask for the influence of external fields on their zero-point oscillations and to investigate its consequences on the Casimir effect. The influence of external fields on zero-point oscillations of unconfined charged bosonic and fermionic fields and the construction of the corresponding effective field theories (at the one-loop level) is an honourable subject and is linked to the pioneering works of the 1930s by Heisenberg, Kockel and Euler, Heisenberg and Euler, and Weisskopf [4] and of the 1950s by Schwinger [5]. More recently, the influence of a uniform magnetic field was extensively treated by Dittrich and Reuter in the monograph [6], and zeta-function techniques were employed to construct effective Lagrangians for scalars and Dirac fields for $d = 2$, 3 and higher dimensions in constant background fields (see [7] and references therein). Ambjorn and Wolfram considered the influence of an electrical field on the vacuum fluctuations of a charged scalar field [8]. Outside the context of pure QED, the influence on the vacuum energy density of a real scalar field due to an arbitrary background of a real scalar field is interpreted as a substitute for hard boundary conditions (see [9–11]). In the gravitational case, Elizalde and Romeo investigated the issue of a neutral scalar field in the presence of a static external gravitational field [12]. A general and valuable discussion of these aspects of quantum field theory can be found in [13] too.

The influence of external fields on zero-point oscillations of quantum fields confined by hard boundary conditions, however, has been investigated in some particular cases only. For example, the influence of a uniform magnetic field on the Casimir effect was investigated in [14] in the cases of a massive fermion field and a charged scalar field both submitted to anti-periodic boundary conditions, and in [15] in the case of a massive charged scalar field submitted to Dirichlet boundary conditions. In the cases considered by these authors it was formally shown that the fermionic Casimir effect is enhanced by the applied magnetic field while the bosonic one is inhibited. In [14, 15], Schwinger's proper-time method [5] was employed and explicit analytical results obtained for particular regimes of a conveniently defined dimensionless parameter μ and magnitude of the external field. In the weak magnetic field regime, some aspects of the material properties of confined charged fields were investigated in [16].

In this work we wish to resume the investigation of the influence of an external magnetic field on the zero-point oscillations of charged confined quantum fields and consider the case of a massive fermion field under a uniform magnetic field and constrained by boundary conditions of the MIT type, which states that the fermionic current through a hypothetical confining surface must be zero [17]. This is not an academic question since, for instance, in the bag model of hadrons [17, 18], we can expect the zero-point oscillations of the quark fields to be influenced by the strong electric and magnetic fields that permeate the interior of the hadronic bag (see, for example, [19] and references therein). We also wish to reconsider the case of a massive charged scalar field under a uniform magnetic field by investigating regimes not considered in [15]. In order to evaluate the relevant Casimir energies, it is convenient, especially in the case of the massive fermion field with MIT boundary conditions, to follow the spirit of the representation of spectral sums as contour integrals [20], which allows for the incorporation of the boundary conditions in a smooth way. Here we will employ a relatively simple variant of this general procedure. This variant is described in [21] and it will be applied here without further explanation to the cases at hand (an interested reader should consult that paper for details). The outline of this work is as follows. In section 2 we apply our calculational tools to the charged scalar field case. In section 3, we consider the case of the charged fermion field. For both cases, detailed numerical analysis and some special analytical limits are provided, together with an explanation of the results obtained and specific comparisons with other results. Section 4 is devoted to final remarks. Throughout the paper we employ natural units ($\hbar = c = 1$).

2. The vacuum energy of a charged scalar field with Dirichlet boundary conditions and under the influence of a uniform magnetic field

Let us first briefly consider a charged scalar field under Dirichlet boundary conditions imposed on the field on two parallel planes separated by the distance ℓ whose side L satisfies the condition $L \gg \ell$. Suppose initially that there is no external field. The unregularized Casimir energy is given by [21][4]

$$E_0(\ell,\mu) = \alpha \frac{L^2}{(2\pi)^3} \int \mathrm{d}^3 p \log\left[1 + \frac{K_1(z)}{K_2(z)}\right] \tag{1}$$

where $K_1(z)$ and $K_2(z)$ are functions constructed from the boundary conditions as described in [21]. The dimensionless parameter α takes into account the internal degrees of freedom of the quantum field. For Dirichlet boundary conditions we begin by writing

$$F(z) = \sin z. \tag{2}$$

Since $z = 0$ is a root of $F(z)$ we divide this function by z, thus removing $z = 0$ from the set of roots without introducing a new singularity: this is equivalent to removing the zero mode. Define

$$G(z) = \frac{\sin z}{z}. \tag{3}$$

The function $K(z) = K_1(z) + K_2(z)$, with $K_1(z) = K_2(-z)$, is then obtained by performing the substitution $z \to \mathrm{i}z$, that is

$$K(z) := G(\mathrm{i}z) = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2z} \tag{4}$$

where z is the function

$$z = z(p_1, p_2, p_3) = \ell\sqrt{p_1^2 + p_2^2 + p_3^2 + m^2}. \tag{5}$$

Since $K_1(z) = -\mathrm{e}^{-z}/2z$ and $K_2(z) = \mathrm{e}^{z}/2z$ we have

$$E_0(\ell,\mu) = \alpha \frac{L^2}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \log[1 - \mathrm{e}^{-2z}]. \tag{6}$$

To shorten these initial steps let us set $m = 0$. Then after expanding the log, we obtain, simply,

$$E_0(\ell) = -\frac{\alpha L^2}{4\pi^2\ell^3} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} \mathrm{d}x\, x^2\, \mathrm{e}^{-2kx} \tag{7}$$

where we have defined $x := p_3\ell$. The integral can be evaluated with the help of the Mellin transform

$$A^{-s}\Gamma(s) = \int \mathrm{d}t\, t^{s-1}\, \mathrm{e}^{-At} \tag{8}$$

and after simple manipulations we obtain the well-known result

$$E_0(\ell) = -\frac{\alpha L^2 \pi^2}{1440\ell^3}. \tag{9}$$

For $\alpha = 2$ we get the result valid for photons. A more complex example of the usefulness of equation (1) is provided in [21].

Let us consider now the same type of field and boundary conditions in the presence of an external magnetic field which we suppose to be uniform and perpendicular to the two Dirichlet

[4] Note that in the absence of external fields we could have started from equation (21) in [21], but for our purposes here it is more convenient to start from equation (18).

planes. We will also assume that eB points towards the positive OX_3 direction. Equation (1) now reads

$$E_0(\ell, \mu, eB) = \alpha \left(\frac{eB}{2\pi}\right) \frac{L^2}{2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_3}{2\pi} \log[1 - \mathrm{e}^{-2z}] \tag{10}$$

where we have taken into account that for a charged spin-zero boson the Landau levels are given by

$$p_1^2 + p_2^2 = eB(2n+1) \qquad n = 0, 1, 2, 3, \ldots. \tag{11}$$

The factor $eB/2\pi$ is the degeneracy factor and for a charged scalar field $\alpha = 2$. Hence, the function z now reads

$$z = z(p_3, n) := \sqrt{\ell^2 p_3^2 + eB\ell^2(2n+1) + \mu^2} \tag{12}$$

where $\mu := \ell m$. It is convenient to define

$$M_n := \sqrt{(2n+1)eB\ell^2 + \mu^2} \tag{13}$$

and write

$$E_0(\ell, \mu, eB) = \frac{eBL^2}{2\pi^2\ell} \sum_{n=0}^{\infty} I_n(M_n) \tag{14}$$

where

$$I_n(M_n) = \int_0^{\infty} \mathrm{d}x \log\left[1 - \mathrm{e}^{-2\sqrt{x^2+M_n^2}}\right] \tag{15}$$

and we have set $x := p_3\ell$. Expanding the log and introducing the variable ω, defined by $\omega := \sqrt{x^2 + M_n^2}$, we end up with

$$E_0(\ell, \mu, eB) = \frac{eBL^2}{2\pi^2\ell} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} I_{kn}(M_n) \tag{16}$$

where

$$I_{kn}(M_n) := \int_{M_n}^{\infty} \mathrm{d}\omega\, \omega (\omega + M_n)^{-1/2} (\omega - M_n)^{-1/2}\, \mathrm{e}^{-2k\omega}. \tag{17}$$

In order to evaluate this integral, we first introduce an auxiliary integral defined by

$$I_{kn}(M_n, \lambda) := \int_{M_n}^{\infty} \mathrm{d}\omega\, (\omega + M_n)^{-1/2} (\omega - M_n)^{-1/2}\, \mathrm{e}^{-2k\omega\lambda}. \tag{18}$$

To evaluate the auxiliary integral we make use of (cf formula 3.384.3 in [22])

$$\int_{\mu_1}^{\infty} \mathrm{d}x\, (x+\beta)^{2\nu-1} (x-\mu_1)^{2\rho-1}\, \mathrm{e}^{-\mu_2 x} = \frac{(\mu_1+\beta)^{\nu+\rho-1}}{\mu_2^{\nu+\rho}} \exp\left[\frac{(\beta-\mu_1)}{2}\mu_2\right] \times \Gamma(2\rho) W_{\nu-\rho,\nu+\rho-\frac{1}{2}}(\mu_1\mu_2 + \beta\mu_2) \tag{19}$$

which holds for $\mu_1 > 0$, $|\mathrm{Arg}\,(\beta+\mu_1)| < \pi$, $\mathrm{Re}\,\mu_2 > 0$ and $\mathrm{Re}\,\rho > 0$. The auxiliary integral then reads

$$I_{kn}(M_n, \lambda) = \frac{(2\mu)^{-1/2}}{(2k\lambda)^{1/2}} \Gamma(1/2) W_{0,0}(4kM_n\lambda) \tag{20}$$

where $W_{\mu,\lambda}(z)$ is the Whittaker function [23]. The Whittaker $W_{0,0}(z)$ function is related to the modified Bessel function of the third kind through [22]

$$W_{0,0}(4kM_n\lambda) = \frac{(4kM_n\lambda)^{1/2}}{\pi^{1/2}} K_0(2kM_n\lambda). \tag{21}$$

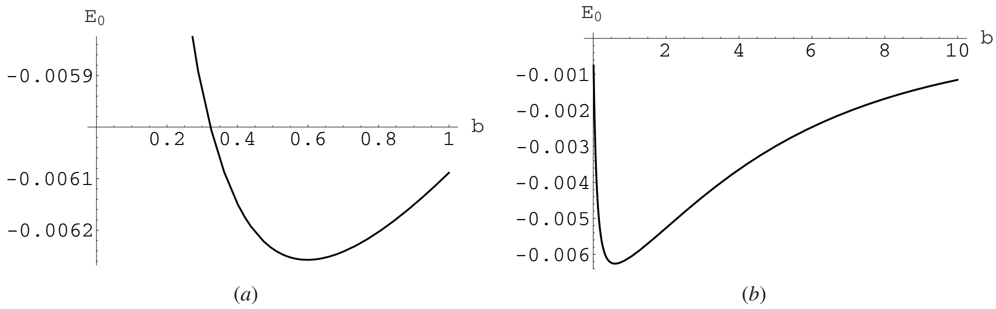

Figure 1. Plot of the energy in terms of the dimensionless quantity $b\,(=eB\ell^2)$ corresponding to the magnetic field, for a fixed value of the dimensionless mass μ (here $\mu = 1$). (*a*) shows in detail the formation of a smooth minimum and its precise value.

Taking the derivative of equation (20) with respect to λ, and setting $\lambda = 1$ to obtain $I_{kn}(M_n)$, we have

$$I_{kn}(M_n) = 2M_n \frac{\mathrm{d}}{\mathrm{d}z} K_0(z = 2kM_n\lambda)|_{\lambda=1} = -M_n K_1(2kM_n). \tag{22}$$

Hence the vacuum energy reads

$$E_0(\ell, \mu, eB) = -\frac{eBL^2}{2\pi^2\ell} \sum_{n=0}^{\infty} M_n \sum_{k=1}^{\infty} \frac{1}{k} K_1(2kM_n) \tag{23}$$

or, more explicitly,

$$\begin{aligned} E_0(\ell, \mu, eB) = &-\frac{eBL^2}{2\pi^2\ell}\sqrt{eB\ell^2+\mu^2}\sum_{k=1}^{\infty}\frac{1}{k}K_1\big(2k\sqrt{eB\ell^2+\mu^2}\big) \\ &-\frac{eBL^2}{2\pi^2\ell}\sum_{n=1}^{\infty}\sqrt{(2n+1)eB\ell^2+\mu^2}\sum_{k=1}^{\infty}\frac{1}{k}K_1\big(2k\sqrt{(2n+1)eB\ell^2+\mu^2}\big). \end{aligned} \tag{24}$$

This representation is an alternative to that obtained in [15]. In a very strong field, such that $eB\ell^2 \gg \mu^2$, the decaying exponential behaviour of the Bessel functions of the third kind allows us to keep only the term corresponding to $k = 1$ in the first parcel of the vacuum energy and, thus, in this limit we have

$$\frac{E_0(\ell, eB)}{L^2} \approx -\frac{(eB\ell^2)^{5/4}}{\pi^{3/2}\ell^3}\, \mathrm{e}^{-2\sqrt{eB\ell^2}} \tag{25}$$

in agreement with [15]. Note that for zero magnetic field the vacuum energy as given by equations (24) and (25) is zero, that is, the zero of the energy is automatically shifted with respect to the vacuum energy in the absence of the external field (which serves, therefore, as the natural origin of energies).

2.1. Numerical analysis for the scalar case

For arbitrary values of the parameter μ and of the scaled field $eB\ell^2$, a numerical evaluation of equation (23) is still possible. The graphs of figures 1(*a*), (*b*), 2 and 3 exhibit some representative examples.

In the first two graphs (figure 1) we plot a dimensionless version of equation (23) as a function of the scaled magnetic field $b := eB\ell^2$, for $\mu = 1$ and a convenient range of b that

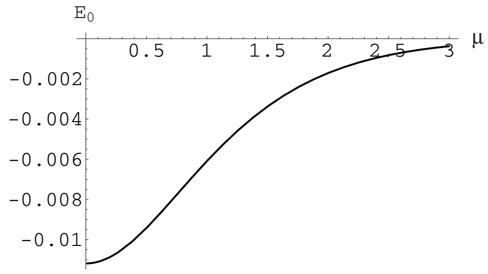

Figure 2. Plot of the energy in terms of the dimensionless mass μ, for a fixed value of the dimensionless magnetic field b (here $b = 1$). For a wide range of values of b one gets a curve with a similar shape.

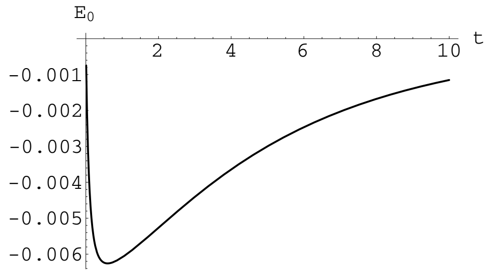

Figure 3. Plot of the energy in terms of the dimensionless variable $t = b/\mu^2$ that measures the magnitude of the dimensionless magnetic field b in terms of the dimensionless mass μ^2. It is clear that at $\mu = 1$ we recover the same shape as in figure 1.

clearly shows the behaviour of the energy (figure 1(*b*)). Note that initially the external field decreases the value of the vacuum energy up to a certain value of b for which the vacuum energy attains a minimum (which is most clearly depicted in figure 1(*a*)). After reaching this point, the energy increases as remarked in [15]. This situation is the opposite of what happens with a fermionic field, in which case the energy decreases linearly with b, for large values of b (as will be clearly seen in the next section).

Figure 2 shows the behaviour of the energy in terms of the mass μ for a fixed value of the magnetic field b (chosen here as $b = 1$, but the shape of the curve is very similar for a wide range of values of b). It starts at a nonzero value, for zero magnetic field. Also this is quite different from the behaviour in the case of a fermionic field.

Finally, in figure 3 the energy is depicted versus the variable $t = b/\mu^2$ that measures the magnitude of the dimensionless magnetic field b in terms of the dimensionless mass μ^2. Note that, although for $\mu = 1$ we obtain the same curve as in figure 1, for any other value of μ the precise shape of figure 3 is valuable, as a way of representing the energy in terms of the unique quantity t (the magnetic field in units of mass squared).

Families of curves corresponding to different values of one of the variables, and also two-dimensional graphs, are easy to obtain in a reasonable amount of time, from our formulae in this section.

3. Confined fermion field in a uniform magnetic field

As in the case of the charged bosons, the uniform magnetic field here is perpendicular to the (hypothetical) parallel MIT constraining surfaces. The distance between them is ℓ and eB is again supposed to point towards the positive OX_3 direction. The starting expression reads

$$E_0(\ell, \mu, eB) = -2 \times \frac{1}{2}\left(\frac{eBL^2}{2\pi}\right)\sum_{n=0}^{\infty}\sum_{\alpha\in\{-1,1\}} I_{n\alpha} \tag{26}$$

where the multiplicative factor 2 takes into account the particle and anti-particle states, and we have defined

$$I_{n\alpha} := \int_{-\infty}^{\infty}\frac{\mathrm{d}p_3}{2\pi}\log\left[1 - \frac{z-\mu}{z+\mu}\,\mathrm{e}^{-2z}\right] \tag{27}$$

where again (with $\mu := \ell m$)

$$z = z(q, n, \alpha) := \sqrt{\ell^2 p_3^2 + (2n+1-\alpha)eB\ell^2 + \mu^2} \qquad n = 0, 1, 2, 3, \ldots. \tag{28}$$

Expanding the log as before, we obtain

$$\log\left[1 - \frac{z-\mu}{z+\mu}\,\mathrm{e}^{-2z}\right] = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}[z+\mu]^{-k}[z-\mu]^k\,\mathrm{e}^{-2kz} \tag{29}$$

and performing the sum over α, we arrive at

$$E_0(\ell, \mu, eB) = -2\frac{eBL^2}{2\pi^2\ell}\sum_{p=-1}^{\infty}{}'\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}I_{pk}(M_p) \tag{30}$$

where

$$I_{pk}(M_p) := \int_0^{\infty}\mathrm{d}x\left[\left(x^2+M_p^2\right)^{1/2}+\mu\right]^{-k}\left[\left(x^2+M_p^2\right)^{1/2}-\mu\right]^{k}\mathrm{e}^{-2k\left(x^2+M_p^2\right)^{1/2}} \tag{31}$$

with $x := p_3\ell$ and

$$M_p^2 := 2(p+1)eB\ell^2 + \mu^2 \qquad p = -1, 0, 1, 2, 3, \ldots. \tag{32}$$

The prime in equation (30) means that the term corresponding to $p = -1$ must be multiplied by the factor 1/2. Let us define, as before, a new variable ω through $\omega = \left(x^2+M_p^2\right)^{1/2}$. Then

$$I_{pk}(M_p) = \int_{M_p}^{\infty}\mathrm{d}\omega\,\omega(\omega+M_p)^{-1/2}(\omega-M_p)^{-1/2}(\omega+\mu)^{-k}(\omega-\mu)^k\,\mathrm{e}^{-2k\omega}. \tag{33}$$

The integral defined by equation (33) is non-trivial. Here we will solve it analytically in two limits, namely: the large- and the small-μ limits. A numerical integration of equation (33), however, is feasible to conveniently complete the analysis. We will show the results later.

3.1. The limit $\mu \ll 1$

If we set $\mu \approx 0$ we have to solve a much simpler integral for $I_{pk}(M_p)$. In fact,

$$I_{pk}(M_p) = \int_{M_p}^{\infty}\mathrm{d}\omega\frac{\omega\,\mathrm{e}^{-2k\omega}}{\sqrt{\omega^2-M_p^2}}. \tag{34}$$

This integral can be evaluated with the help of (cf formula 3.365.2 of [22])

$$\int_a^{\infty}\frac{x\,\mathrm{e}^{-bx}}{\sqrt{x^2-a^2}} = aK_1(ab) \qquad a > 0 \quad \mathrm{Re}\,a > 0. \tag{35}$$

For $a = 0$ this integral reads

$$\int_0^{\infty}\mathrm{e}^{-bx} = \frac{1}{b} \tag{36}$$

as can easily be seen by recalling that, for $z \to 0$, $K_1(z) \approx 1/z$. There are three types of contributions to the vacuum energy: corresponding to $p = -1$, $p = 0$ and $p = 1, 2, 3, \ldots$, respectively. They read

$$E_0(\ell, \mu \approx 0, eB) = -\frac{eBL^2}{2\pi^2 \ell} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} I_{-1k} - \frac{eBL^2}{\pi^2 \ell} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} I_{0k}$$
$$- \frac{eBL^2}{\pi^2 \ell} \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} I_{pk} \tag{37}$$

where the integrals in the partial sums are given by

$$I_{-1k} = \frac{1}{2k} \tag{38}$$

$$I_{0k} = \sqrt{2eB\ell^2} K_1\big(2k\sqrt{2eB\ell^2}\big) \tag{39}$$

$$I_{pk} = \sqrt{2(p+1)eB\ell^2} K_1\big(2k\sqrt{2(p+1)eB\ell^2}\big). \tag{40}$$

It follows then that the first sum can be exactly evaluated by using the eta function, $\eta_R(z) = (1 - 2^{1-s})\zeta_R(s)$. In fact, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} I_{-1k} = \frac{\pi^2}{24}. \tag{41}$$

Therefore, the Casimir energy in this limit reads

$$E_0(\ell, \mu \approx 0, eB) = -\frac{eBL^2}{48\ell} - \frac{eBL^2}{\pi^2 \ell} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sqrt{2eB\ell^2} K_1\big(2k\sqrt{2eB\ell^2}\big)$$
$$- \frac{eBL^2}{\pi^2 \ell} \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sqrt{2(p+1)eB\ell^2} K_1\big(2k\sqrt{2(p+1)eB\ell^2}\big). \tag{42}$$

From this result we can easily extract the limit $eB\ell^2 \gg 1$. Due to the decaying exponential behaviour of the modified Bessel function, the leading contribution is given by the first term in equation (42). Hence

$$E_0(\ell, \mu \approx 0, eB) \approx -\frac{eBA}{48\ell} \qquad eB\ell^2 \gg 1. \tag{43}$$

This result has the same sign—and one fourth of the magnitude—of the corresponding problem with anti-periodic boundary conditions in this same limit [14]. This difference can be readily understood if we recall that the anti-periodicity length corresponds to 2ℓ, and that the number of allowed modes is twice the number of modes of the MIT case for massless fermions. Therefore, the result given by equation (43), in that particular limit, is compatible with that given in [14].

3.2. The limit $\mu \gg 1$

In this limit only the term corresponding to $p = -1$, $M_{-1} = \mu$, contributes. Therefore, we have to calculate

$$I_{-1k}(\mu) = \int_{\mu}^{\infty} \mathrm{d}\omega\, \omega (\omega + \mu)^{-k-1/2} (\omega - \mu)^{k-1/2}\, \mathrm{e}^{-2k\omega}. \tag{44}$$

We have learned above the trick to evaluate this integral. One should write the auxiliary integral

$$I_{-1k}(\mu,\lambda) := \int_{\mu}^{\infty} \mathrm{d}\omega\,(\omega+\mu)^{-k-1/2}(\omega-\mu)^{k-1/2}\,\mathrm{e}^{-2k\omega\lambda} \tag{45}$$

and evaluate it with the help of the formula given by equation (19). The result is

$$I_{-1k}(\mu,\lambda) = \frac{\Gamma(k+1/2)}{(4\mu k\lambda)^{1/2}} W_{-k0}(4\mu k\lambda). \tag{46}$$

For large values of the argument, the Whittaker function behaves as

$$W_{\nu\mu}(z) \approx z^{\nu}\,\mathrm{e}^{-z/2} \tag{47}$$

hence for $\mu \gg 1$ we can write

$$I_{-1k}(\mu,\lambda) \approx \Gamma(k+1/2)\frac{\mathrm{e}^{-2\mu k\lambda}}{(4\mu k\lambda)^{k+1/2}} \tag{48}$$

and since

$$I_{-1k}(\mu,\lambda) = -\frac{1}{2k}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\Gamma(k+1/2)\frac{\mathrm{e}^{-2\mu k\lambda}}{(4\mu k\lambda)^{k+1/2}}\right]_{\lambda=1} \tag{49}$$

we finally have

$$I_{-1k}(\mu,\lambda) = \frac{\Gamma(k+1/2)}{2^{2k+2}k^{k+3/2}}\left[\frac{(k+1/2)}{\mu^{k+1/2}} + \frac{2k}{\mu^{k-1/2}}\right]\mathrm{e}^{-2\mu k}. \tag{50}$$

The vacuum energy in this regime is given by

$$E_0(\ell,\mu\gg 1, eB) \approx -\frac{eBL^2}{8\pi^2\ell}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2^{2k}k^{k*5/2}}\Gamma(k+1/2)\left[\frac{(k+1/2)}{\mu^{k+1/2}} + \frac{2k}{\mu^{k-1/2}}\right]\mathrm{e}^{-2\mu k} \tag{51}$$

a result that is new and incorporates the first mass corrections.

The most relevant one is the $k = 1$ term, which yields

$$E_0(\ell,\mu\gg 1, eB) \approx -\frac{eBL^2}{32\pi^{3/2}\ell}\frac{\mathrm{e}^{-2\mu}}{\mu^{1/2}} \tag{52}$$

which is sufficiently small. This is not the same result as that obtained for anti-periodic boundary conditions though the damping exponential appears in both cases. The factors multiplying the exponential are different, it must be remembered though that for $\mu \neq 0$ the MIT and the AP spectra are not comparable. When $\mu = 0$ the MIT and the AP spectra for each component of the fermion field differ by a factor of 4.

3.3. Numerical analysis for arbitrary mass

For arbitrary values of the mass and the magnetic field or, correspondingly, of their dimensionless counterparts, μ and $eB\ell^2$, the integral defined by (33) is very difficult to evaluate analytically. In fact, even if we give it in terms of hypergeometric functions this would not improve our knowledge of the dependence of the energy on the mass and the magnetic field. Fortunately, a numerical analysis based on equation (30) is possible and leads to very precise, easily understandable results.

To this end, first we rewrite equation (30) in the form

$$\frac{\ell^3 E_0(t,\mu)}{L^2} = -\frac{t\mu^3}{\pi^2}\sum_{p=-1}^{\infty}{}'\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}J_{pk}(t,\mu) \tag{53}$$

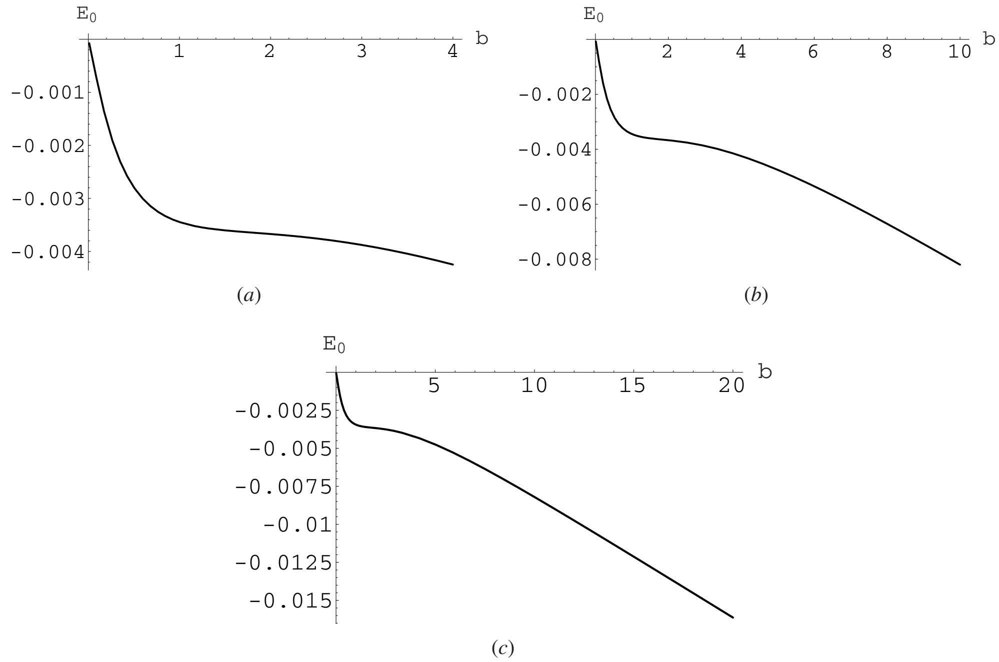

Figure 4. In the fermionic case, a plot of the energy in terms of the dimensionless magnetic field b, for a fixed value of the dimensionless mass μ (here $\mu = 1$). (*a*) Details are shown of the formation of an inflection region and the inflection point, (*b*) shows the intermediate region and finally, (*c*) shows asymptotic behaviour for large values of b.

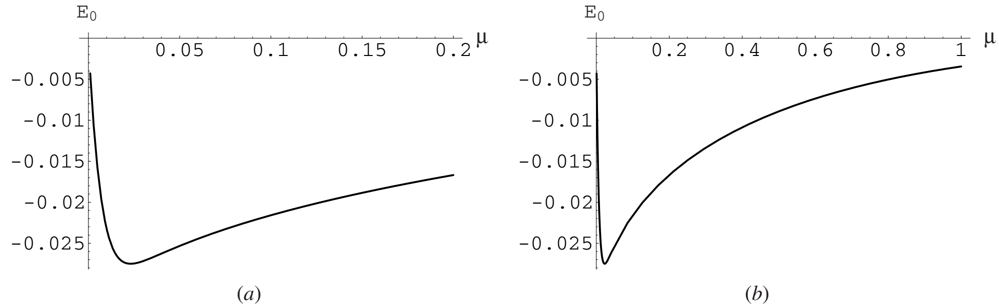

Figure 5. Plot of the energy in terms of the dimensionless mass μ. (*a*) The behaviour for small mass is shown with the formation of a smooth minimum, which looks much sharper in (*b*), where the asymptotic behaviour for large μ is clearly established.

where

$$J_{pk}(t,\mu) := \int_0^\infty \mathrm{d}y\,[(y^2+2(p+1)t+1)^{1/2}+1]^{-k}[(y^2+2(p+1)t+1)^{1/2}-1]^k \times \mathrm{e}^{-2k\mu(y^2+2(p+1)t+1)^{1/2}} \tag{54}$$

with $y := x/\mu$ and $t = eB\ell^2/\mu^2$.

The result of the numerical evaluation of equation (54) for a sample of values of μ and b is shown in figures 4 and 5. Figure 4 shows the dependence of the energy on the magnetic field b for different ranges of this quantity, to better show the inflection region (figure 4(*a*), low values of μ), the intermediate region (figure 4(*b*)) and the asymptotic behaviour for large values of b

(figure 4(*c*)). The intermediate region is most interesting, showing a smooth transition from the case of small magnetic field (with a good zero field limit) to the linear asymptotic behaviour corresponding to a large field. And this situation is common for any particular value of the mass μ. The transition is given by a smooth inflection point of nearly horizontal tangent. It looks like a minimum could almost be formed, but not quite. In figure 5 the dependence of the energy on the dimensionless mass μ is depicted, showing the behaviour in the region of small mass (figure 5(*a*)) and the asymptotic behaviour for large μ (figure 5(*b*)). Needless to say, the asymptotic behaviour corresponds to the analytic expressions obtained before. For the intermediate region we see a smooth behaviour connecting the two regions with a minimum that is very easily obtained with good precision, for any particular value of b.

The same observations as for the bosonic case can be made here, namely, that the graphs shown are only a very small sample of those that can easily be obtained from our general formulae in this section, corresponding to the fermionic case, for different values of the variables, and including two-dimensional plots. We should warn the reader, however, that the computation time is now drastically increased.

4. Final remarks

In this paper we have considered the influence of an external uniform magnetic field on the Casimir energy associated with charged quantum fields confined by hard boundary conditions. The method employed incorporates, in a relatively simple way, the hard boundary conditions of the Dirichlet or MIT type imposed, respectively, on a massive bosonic field and a massive fermionic field plus, for both cases, the effect of the magnetic field, and leads to expressions for the vacuum energy especially suited for numerical calculations. The analytical and numerical results show that, in the bosonic case and Dirichlet boundary conditions, the effect of a large, applied magnetic field is to suppress the vacuum energy, and in the case of a fermionic field confined in a slab-bag with MIT boundary conditions, the effect of the applied field is to enhance (in absolute magnitude) the vacuum energy. For small and intermediate values of the scaled magnetic field, however, a more interesting behaviour of the vacuum energy shows up. Since here the calculations were performed in the general framework of an effective field theory, the details of the interaction between the quantum field and the external magnetic field—represented by the generation of the Landau levels and the hard boundary conditions—remain hidden. It would be interesting to understand the behaviour of the vacuum energy under the circumstances considered here in terms of a more fundamental model of the structure of the confined quantum vacuum.

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